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Line singularities in unbounded stratified flow

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We obtain the solutions, under the Oseen and Boussinesq approximations, for the flow field disturbance due to a line singularity in an otherwise uniform, horizontal, inviscid, incompressible flow of a vertically stratified fluid. The results obtained show no upstream influence for those singularities across which $\nabla^2\psi + (N/U)^2\psi$ is continuous. Doublet and vortex singularities are examples of these. Uniform flows past doublets and vortices are considered for a range of internal Froude numbers, including the calculation of the pressure distributions and drag for the doublet. An application of the vortex solution to flows in the β-plane is discussed.

1. Introduction

Singular solutions to Laplace's equation in two dimensions are used extensively to obtain irrotational flows over two-dimensional obstacles. We seek here solutions for the analogues in stratified flow of the irrotational doublet and vortex as well as other line singularities. We initially seek singular solutions which, away from the origin, are weak perturbations on a uniform horizontal flow of a fluid whose (conserved) density field decreases linearly with height. Those singularities across which $\nabla^2\psi + (N/U)^2\psi$ is continuous are shown to be consistent with no upstream influence and their solutions may be regarded as solutions, under the Boussinesq approximation, of Long's model. Therefore, for these singularities the restriction that the perturbation be small is lifted. The solutions are obtained in integral form and so differ from the solution obtained by Miles (1968) for stratified flow past a circular cylinder. Miles' solution is in the form of an infinite series of cylindrical lee-wave functions which are themselves composed of an infinite series of products of Bessel and trigonometric functions. Miles' solution apparently decays more rapidly upstream than that obtained here. The integral form of our solutions is well suited for superposition, which will also be discussed.

2. Specification of the problem

We consider a slightly disturbed, incompressible, horizontal, linearly stratified, inviscid flow. The perturbation (unprimed) quantities are related to the total (primed) quantities as follows:

$$\left.\begin{aligned} u' &= U+u = -\partial(-Uz+\psi)/\partial z, \quad w' = w = \partial\psi/\partial x, \\ \psi' &= -Uz+\psi, \quad \rho' = \rho_0[1-\beta z+(\beta/U)\psi], \\ p' &= p_0-\rho_0 gz+\tfrac{1}{2}\rho_0\beta g z^2+p. \end{aligned}\right\} \qquad (1)$$

For a small perturbation, the equation governing the behaviour of u under the Boussinesq approximation is as follows:

$$\partial(\nabla^2 u)/\partial x + \alpha^2\, \partial u/\partial x = 0, \tag{2}$$

where $\alpha^2 \equiv (N/U)^2 = \beta g/U^2$. We seek solutions of (2) which are even (E) or odd (O) functions of z of the form

$$\left.\begin{matrix} u_E(x,z) \\ u_O(x,z) \end{matrix}\right\} = \int_0^\infty F(K,x) \begin{Bmatrix} \cos Kz \\ \sin Kz \end{Bmatrix} dK. \tag{3}$$

Substituting (3) into (2) and following Janowitz (1973) we find that

$$u^-_{E,O} = \int_0^\alpha A_{E,O}(K)\,(c,s)\,dK + \int_\alpha^\infty A_{E,O} \exp\,[(K^2-\alpha^2)^{\frac{1}{2}} x]\,(c,s)\,dK \quad \text{for} \quad x \leqslant 0, \tag{4a}$$

$$u^+_{E,O} = \int_\alpha^\infty B_{E,O}(K)\,(c,s)\,dK + \int_\alpha^\infty C_{E,O} \exp\,[-(K^2-\alpha^2)^{\frac{1}{2}} x]\,(c,s)\,dK$$
$$+ \int_0^\alpha \{B_{E,O} \cos\,[(\alpha^2-K^2)^{\frac{1}{2}} x] + C_{E,O} \sin\,[(\alpha^2-K^2)^{\frac{1}{2}} x]\}\,(c,s)\,dK \quad \text{for} \quad x \geqslant 0, \tag{4b}$$

where $$(c,s) \equiv (\cos Kz, \sin Kz).$$

We shall require that the upstream and downstream solutions for u, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ be continuous at $x = 0$ for all $z > 0$. The nature of the discontinuity in u, $\partial u/\partial x$ or $\partial^2 u/\partial x^2$ at $z = 0$ will determine the nature of the flow disturbance. We shall make extensive use of Fourier transforms. All the necessary formulae are present in Erdélyi *et al.* (1954).

3. The general singularity

We now temporarily drop the subscripts and include the singularity as a discontinuity at the origin as follows:

$$u^-(0,z) - u^+(0,z) = H_0(z), \tag{5a}$$

$$\partial u^-(0,z)/\partial x - \partial u^+(0,z)/\partial x = H_1(z), \tag{5b}$$

$$\partial^2 u^-(0,z)/\partial x^2 - \partial^2 u^+(0,z)/\partial x = H_2(z), \tag{5c}$$

where the $H_n(z)$ $(n = 0, 1, 2)$ are functions which vanish for $z > 0$. We define $H_n(K)$ as follows:

$$H_n(z) = \int_0^\infty H_n(K)\,(c,s)\,dK. \tag{6}$$

Substituting (4) and (6) into (5) leads to the following:

$$\left.\begin{aligned} A(K) - B(K) &= H_0(K) \quad \text{for} \quad K < \alpha, \\ A(K) - B(K) - C(K) &= H_0(K) \quad \text{for} \quad K > \alpha, \end{aligned}\right\} \tag{7a}$$

$$\left.\begin{aligned} -(\alpha^2-K^2)^{\frac{1}{2}}\, C(K) &= H_1(K) \quad \text{for} \quad K < \alpha, \\ (K^2-\alpha^2)^{\frac{1}{2}}\,(A(K)+C(K)) &= H_1(K) \quad \text{for} \quad K > \alpha, \end{aligned}\right\} \tag{7b}$$

$$\left.\begin{aligned} -(K^2-\alpha^2)\,B(K) &= H_2(K) \quad \text{for} \quad K < \alpha, \\ (K^2-\alpha^2)\,(A(K)-C(K)) &= H_2(K) \quad \text{for} \quad K > \alpha. \end{aligned}\right\} \tag{7c}$$

Solving these equations for A, B and C we find

$$u^-(x,z) = \int_0^\alpha \left(H_0 - \frac{H_2}{K^2-\alpha^2}\right)(c,s)\,dK$$
$$+\int_\alpha^\infty \left(\frac{H_2}{2(K^2-\alpha^2)} + \frac{H_1}{2(K^2-\alpha^2)^{\frac{1}{2}}}\right)\exp\left[(K^2-\alpha^2)^{\frac{1}{2}}x\right](c,s)\,dK \quad \text{for} \quad x \leqslant 0, \qquad (8a)$$

$$u^+(x,z) = \int_\alpha^\infty \left(\frac{H_2}{K^2-\alpha^2} - H_0\right)(c,s)\,dK$$
$$-\int_0^\alpha \left(\frac{H_2\cos\left[(\alpha^2-K^2)^{\frac{1}{2}}x\right]}{K^2-\alpha^2} + \frac{H_1\sin\left[(\alpha^2-K^2)^{\frac{1}{2}}x\right]}{(\alpha^2-K^2)^{\frac{1}{2}}}\right)(c,s)\,dK$$
$$+\int_\alpha^\infty \left(\frac{H_1}{2(K^2-\alpha^2)^{\frac{1}{2}}} - \frac{H_2}{2(K^2-\alpha^2)}\right)\exp\left[-(K^2-\alpha^2)^{\frac{1}{2}}x\right](c,s)\,dK \quad \text{for} \quad x \geqslant 0. \qquad (8b)$$

From (8) and (9), below, the following conclusion may be drawn. If the quantity $\nabla^2\psi + (N/U)^2\psi$ ($\equiv L\psi$) is continuous across the singularity, then there is no upstream influence; the resulting solution is thus a solution of Long's model, under the Boussinesq approximation (§ 5), and is valid at all points for all magnitudes of the disturbance. This follows from the fact that if $L\psi$ is continuous then $H_0(K) - H_2(K)/(K^2-\alpha^2)$ is identically zero; the columnar disturbances in (8) then drop out of the solution. On the other hand, if $L\psi$ is discontinuous, then upstream influence occurs and $u^+(x,0)$ is singular for all $x \geqslant 0$; upstream influence in a vertically unbounded domain is apparently produced only by an obstacle which is unbounded in the downstream direction. The quantity $L\psi$ can be interpreted as 'potential vorticity' composed of the vorticity and 'available' potential vorticity. If the singularity is a line source of this 'potential' vorticity then upstream influence occurs.

We now specify the general form of the $H_n(K)$. If the $H_n(z)$ were non-zero for $0 \leqslant z < \delta$, then their transforms would be analytic and expandable in power series in K; in our case $\delta \to 0$. Therefore, for $u_E(x,z)$ ($u_O(x,z)$) we take $H_n(K)$ to be a polynomial in K^{2m} (K^{2m+1}) for $m = 0, 1, 2, \ldots$. This is possible since, for $z > 0$,

$$\int_0^\infty K^{2m}\begin{Bmatrix}\cos Kz \\ K\sin Kz\end{Bmatrix}dK \equiv \lim_{\epsilon\to 0}\int_0^\infty K^{2m}e^{-K\epsilon}\begin{Bmatrix}\cos Kz \\ K\sin Kz\end{Bmatrix}dK = 0.$$

Now ψ, $\partial\psi/\partial x$ and $\partial^2\psi/\partial x^2$ will be obtained by integrating (8) with respect to z and then differentiating with respect to x. For these functions to be continuous for $z > 0$ we further require that

$$H_1(0) = H_2(0) = H_2(\alpha) = 0. \qquad (9)$$

Before proceeding to the solution for the doublet and vortex we obtain, as a special case, the source solution of Wong & Kao (1970) by taking $H_0 = -Q$, $H_1 = H_2 = 0$ and $\cos Kz$ in (8). This yields

$$u(x,z) = -Q\sin(\alpha z)/z + U(x)\,Q\delta(z).$$

Since $H_0 - H_2/(K^2-\alpha^2)$ is not zero ($= -Q$), upstream influence is present; this solution may be regarded as a far-field solution for small Q.

We now proceed to two singularities which by their near-field behaviour do not produce upstream influence and hence give solutions valid at all points for all magnitudes.

4. The doublet and vortex singularities

We specify the doublet and vortex singularities in the stratified case by requiring that their behaviour in the neighbourhood of the singularity be identical with that of their irrotational analogues. For a potential doublet of strength P aligned along the x axis

$$u^I(x,z) = P(x^2-z^2)/(x^2+z^2) = P\int_0^\infty K\,e^{-K|x|}\cos Kz\,dK. \tag{10}$$

We require that $u^{\pm}(0,z)\to u^I(0,z)$ as $z\to 0$. This implies that their transforms are identical at $x=0$ for $K\to\infty$. Setting $x=0$ in (8), choosing $\cos Kz$ and $H_n = H_n(K^2)$, and letting $K\to\infty$, we require

$$H_2/2(K^2-\alpha^2)+H_1/2(K^2-\alpha^2)^{\frac{1}{2}}\to PK,$$
$$H_2/2(K^2-\alpha^2)+H_1/2(K^2-\alpha^2)^{\frac{1}{2}}-H_0\to PK.$$

Therefore,we take

$$H_0 = 0,\quad H_1 = 2PK^2,\quad H_2 = 0. \tag{11}$$

Substituting this into (8) yields

$$u^-(x,z) = P\int_\alpha^\infty \frac{K^2}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[(K^2-\alpha^2)^{\frac{1}{2}}x\right]\cos Kz\,dK, \tag{12a}$$

$$u^+(x,z) = P\int_\alpha^\infty \frac{K^2}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[-(K^2-\alpha^2)^{\frac{1}{2}}x\right]\cos Kz\,dK - 2P\int_0^\alpha \frac{K^2}{(\alpha^2-K^2)^{\frac{1}{2}}}\sin\left[(\alpha^2-K^2)^{\frac{1}{2}}x\right]\cos Kz\,dK. \tag{12b}$$

Integrating these with respect to z leads to

$$\psi^-(x,z) = -P\int_\alpha^\infty \frac{K}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[(K^2-\alpha^2)^{\frac{1}{2}}x\right]\sin Kz\,dK, \tag{12c}$$

$$\psi^+(x,z) = -P\int_\alpha^\infty \frac{K}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[-(K^2-\alpha^2)^{\frac{1}{2}}x\right]\sin Kz\,dK + 2P\int_0^\alpha \frac{K}{(\alpha^2-K^2)^{\frac{1}{2}}}\sin\left[(\alpha^2-K^2)^{\frac{1}{2}}x\right]\sin Kz\,dK. \tag{12d}$$

Equations (12) are our solution for a doublet of strength P at $(0,0)$. We note that a uniform flow of speed U always accompanies this solution. The perturbations vanish as $x\to-\infty$ and these solutions may be regarded as solutions to Long's model under the Boussinesq approximation (§ 5). This implies that the requirement that the perturbation be small may be dropped.

We can show that for $-\alpha x \gg 1$ and $(\alpha x)^2 \gg \alpha z$

$$\psi^-(x,z)\to -P\sin(\alpha z)/x, \tag{13a}$$
$$u^-(x,z)\to \alpha P\cos(\alpha z)/x. \tag{13b}$$

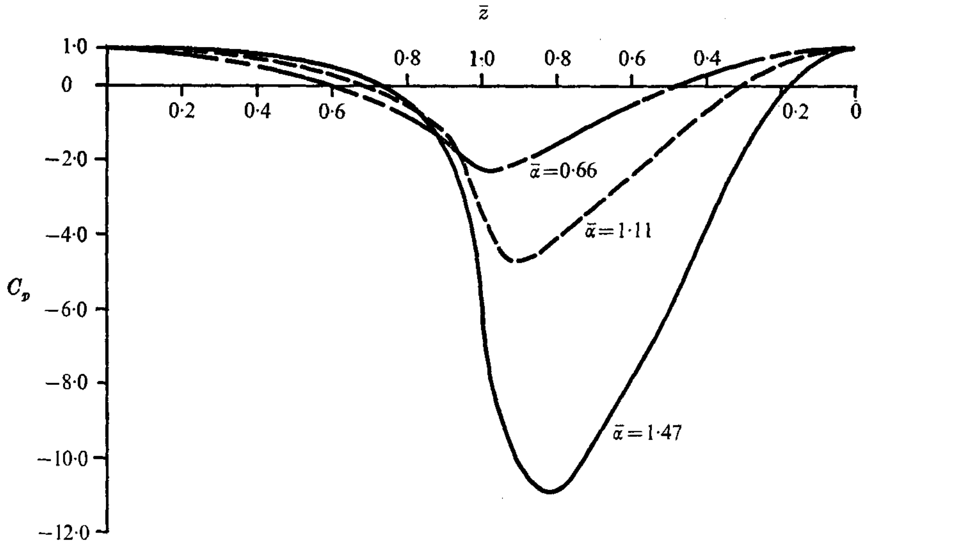

FIGURE 1. The pressure distribution on the surface of the cylinder for various values of $\bar{\alpha}$, where $C_p = 2(p'+\rho_0 gz - p_0)/\rho_0 U^2$, $\bar{z} = z/H$, $\bar{\alpha} = NH/U$.

Miles' (1968) solution requires that upstream the perturbation stream function decay more rapidly than $r^{-\frac{1}{2}}$. Hence the perturbation radial velocity should decay more rapidly than $r^{-\frac{3}{2}}$, which, at $z = 0$, differs from (13*b*). To discuss the flow past a circular cylinder we consider a doublet of strength $-P$ in the uniform flow. We define D as $(P/U)^{\frac{1}{2}}$ and define γ as ND/U. We consider the flow for $\gamma = 0{\cdot}5$, $1{\cdot}0$, $1{\cdot}4$ and $2{\cdot}0$. The stream function and the velocity field are computed from (12). The body streamline differs little from a circular cylinder. As α increases, the body becomes slightly elongated in the flow direction. For $\gamma = 0{\cdot}5$, $1{\cdot}0$, $1{\cdot}4$ and $2{\cdot}0$, the height H of the body streamline is equal to $1{\cdot}10D$, $1{\cdot}11D$, $1{\cdot}05D$ and $0{\cdot}93D$. The pressure distribution along the body streamline is obtained from the Bernoulli equation along the body streamline:

$$p' + \rho_0 gz + \tfrac{1}{2}\rho_0(u'^2 + w'^2) = p_0 + \tfrac{1}{2}\rho_0 U^2.$$

The drag per unit width is obtained by integrating the pressure distribution over the body streamline for $z \geqslant 0$. We define pressure and drag coefficients and $\bar{\alpha}$ as follows:

$$C_p = 2(p' + \rho_0 gz - p_0)/\rho_0 U^2,$$
$$C_D = 2D'/\rho_0 U^2 H,$$
$$\bar{\alpha} = NH/U.$$

In figure 1, we plot C_p *vs.* $\bar{z}$ $(= z/H)$ for a range of $\bar{\alpha}$. In this figure $\bar{z}$ increases from zero at the upstream stagnation point to one at the top then decreases to zero at the downstream stagnation point. In figure 2, we plot C_D *vs.* $\bar{\alpha}$. The drag coefficient obtained by Miles coincides with our results for $\bar{\alpha} \leqslant 1{\cdot}3$. The flow patterns obtained are quite similar to Miles' and will not be presented here. We note that our results show that reversed flow occurs in the lee-wave field at some value of $\bar{\alpha}$ between 1·11 and 1·47. Miles fixed this value at 1·27. At larger values of $\bar{\alpha}$, owing to static instability the results presented in the figures may not apply.

The flow disturbance due to a continuous distribution of doublets along the

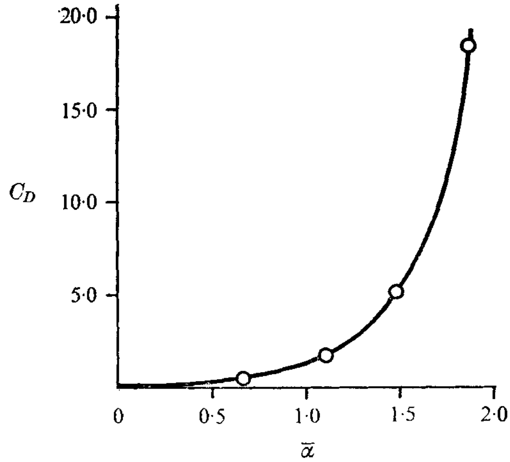

FIGURE 2. The drag coefficient of a semicircular cylinder for various values of $\bar{\alpha}$, where $C_D = 2D'/\rho_0 U^2 H$.

x axis may be easily obtained using (12). Consider a continuous distribution of doublets with density $P'(x_0)$ on the x axis between $x_0 = -L$ and $x_0 = 0$. Using (12c, d) and defining $\overline{\psi}^{\pm} = \psi^{\pm}|_{P=1}$ we find

$$\psi' = -Uz + \int_{-L}^{0} P'(x_0)\,\overline{\psi}^{-}(x-x_0, z)\,dx_0 \quad \text{for} \quad x < -L,$$

$$\psi' = -Uz + \int_{-L}^{x} P'(x_0)\,\overline{\psi}^{+}(x-x_0, z)\,dx_0 + \int_{x}^{0} P'(x_0)\,\overline{\psi}^{-}(x-x_0, z)\,dx_0 \quad \text{for} \quad -L < x < 0,$$

$$\psi'(x,z) = -Uz + \int_{-L}^{0} \overline{\psi}^{+}(x-x_0, z)\,P'(x_0)\,dx_0 \quad \text{for} \quad x > 0.$$

For some cases, the integration with respect to x_0 can be carried out before that with respect to K. For example, if $P' = -P/L$, the stream function becomes

$$\psi' = -Uz + \frac{P}{L}\int_0^{\infty} (e^{t(x+L)} - e^{tx}) \sin[(\alpha^2+t^2)^{\frac{1}{2}} z]\frac{dt}{t} \quad \text{for} \quad x < -L, \tag{14a}$$

$$\psi' = -Uz + \frac{P}{L}\int_0^{\infty} (2 - e^{tx} - e^{-t(x+1)}) \sin[(\alpha^2+t^2)^{\frac{1}{2}} z]\frac{dt}{t} - \frac{2P}{L}\int_0^{\alpha} \{1 - \cos[t(x+L)]\} \sin[(\alpha^2+t^2)^{\frac{1}{2}} z]\frac{dt}{t} \quad \text{for} \quad -L < x < 0, \tag{14b}$$

$$\psi' = -Uz + \frac{P}{L}\int_0^{\infty} (e^{-tx} - e^{-t(x+L)}) \sin[(\alpha^2+t^2)^{\frac{1}{2}} z]\frac{dt}{t} - \frac{2P}{L}\int_0^{\alpha} \{\cos tx - \cos[t(x+L)]\} \sin[(\alpha^2+t^2)^{\frac{1}{2}} z]\frac{dt}{t} \quad \text{for} \quad x > 0, \tag{14c}$$

where we have introduced the integration variable $t = |K^2 - \alpha^2|$. The body streamline is approximately a rectangle of height $\pi P/UL$ and width L for small αL. We next consider the disturbance due to a line vortex.

For the potential line vortex

$$u^I(x,z) = -\Gamma z/2\pi(x^2+z^2) = -\frac{\Gamma}{2\pi}\int_0^\infty e^{-K|x|}\sin Kz\,dK.$$

The stratified analogue is obtained by taking $\sin Kz$ and $H_n = H_n(K^{2m+1})$ in (8), setting $x = 0$ and requiring that the behaviour of the Fourier transforms be identical for large K. We find

$$H_2/2(K^2-\alpha^2)+H_1/2(K^2-\alpha^2)^{\frac{1}{2}} \to -\Gamma/2\pi,$$

$$H_2/2(K^2-\alpha^2)+H_1/2(K^2-\alpha^2)^{\frac{1}{2}} - H_0 \to -\Gamma/2\pi.$$

Hence, $$H_0 = 0, \quad H_1 = -\Gamma K/\pi, \quad H_2 = 0.$$

Thus, for the vortex,

$$u^-(x,z) = -\frac{\Gamma}{2\pi}\int_\alpha^\infty \frac{K}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[(K^2-\alpha^2)^{\frac{1}{2}}x\right]\sin Kz\,dK, \tag{15a}$$

$$u^+(x,z) = -\frac{\Gamma}{2\pi}\int_\alpha^\infty \frac{K}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[-(K^2-\alpha^2)^{\frac{1}{2}}x\right]\sin Kz\,dK + \frac{\Gamma}{\pi}\int_0^\alpha \frac{K}{(\alpha^2-K^2)^{\frac{1}{2}}}\sin\left[(\alpha^2-K^2)^{\frac{1}{2}}x\right]\sin Kz\,dK. \tag{15b}$$

Integrating these functions with respect to z yields, for the vortex,

$$\psi^-(x,z) = -\frac{\Gamma}{2\pi}\int_\alpha^\infty \frac{1}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[(K^2-\alpha^2)^{\frac{1}{2}}x\right]\cos Kz\,dK, \tag{15c}$$

$$\psi^+(x,z) = \frac{\Gamma}{\pi}\int_0^\alpha \frac{\sin\left[(\alpha^2-K^2)^{\frac{1}{2}}x\right]}{(\alpha^2-K^2)^{\frac{1}{2}}}\cos Kz\,dK - \frac{\Gamma}{2\pi}\int_\alpha^\infty \frac{1}{(K^2-\alpha^2)^{\frac{1}{2}}}\exp\left[-(K^2-\alpha^2)^{\frac{1}{2}}x\right]\cos Kz\,dK. \tag{15d}$$

We note that again the columnar disturbance vanishes and we may regard our solutions as solutions to Long's model under the Boussinesq approximation (§ 5).

We note that both the doublet and vortex solutions decay vertically as $(\alpha z)^{-\frac{1}{2}}$. For the Boussinesq approximation to be valid the disturbance must decay in distances small compared with the scale height $1/\beta$. This requires that $\alpha/\beta = g/UN \gg 1$. This condition is well satisfied by oceanic flows and somewhat less so by atmospheric flows.

The size D of a potential vortex in a uniform flow, which may be defined as the distance from the singularity to the stagnation point, is $|\Gamma|/2\pi U$. An appropriate internal Froude number $(U/ND = 1/\alpha D)$ is $2\pi U/|\Gamma|\alpha$. In figure 3, we plot the flow past a vortex with (*a*) $F_i = 0{\cdot}667$ and (*b*) $F_i = 1{\cdot}0$. In each case the circulation is negative: distances have been non-dimensionalized with respect to $1/\alpha$ and the total stream function with respect to U/α.

We observe that (2) also governs the perturbation to a horizontal, homogeneous, eastward current of uniform depth H in the β-plane, in the absence of Ekman suction; we interpret (α^2, z, w) as $(\beta/U, y, v)$, where y and v are the northward variables, f the Coriolis parameter and, in this context, $\beta = df/dy$. If a bump of volume V and horizontal dimension L $[\ll (U/\beta)^{\frac{1}{2}}]$ were located at the bottom

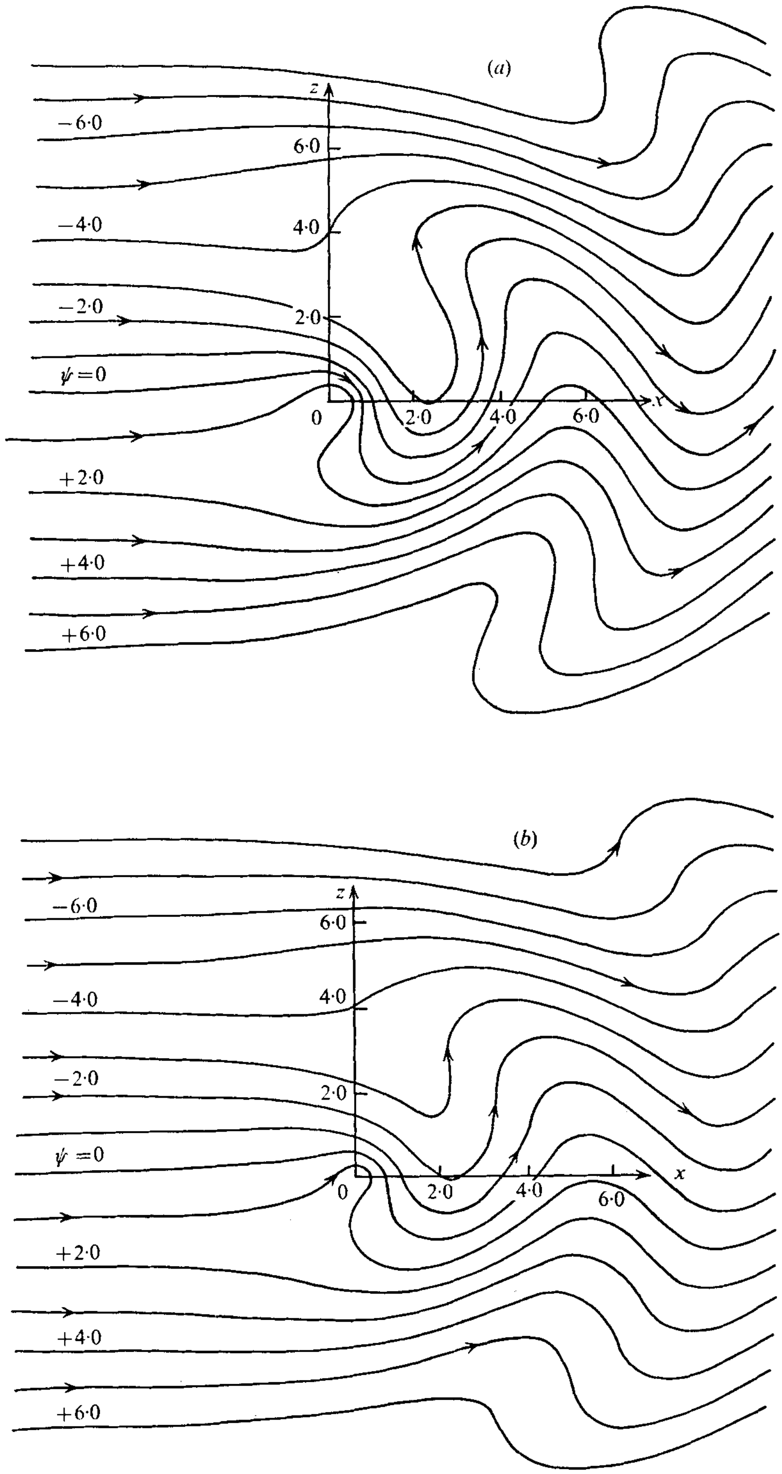

FIGURE 3. Flow past a vortex with $(x, z, \psi) \equiv \alpha(x', z', \psi'/U)$. (a) $2\pi U/\Gamma\alpha = -0{\cdot}67$. (b) $2\pi U/\Gamma\alpha = -1{\cdot}00$.

of the layer, about the origin, and if the flow away from the bump were determined primarily by the circulation induced by the bump, which would be of order $-fV/H$, then the vortex solution and figure 3 would characterize such a flow. In reality, Ekman suction would cause this disturbance to disappear at a distance L_D of order $U/fE^{\frac{1}{2}}$, where E is the vertical Ekman number. Our solution would be meaningful in this context if $(U/\beta)^{\frac{1}{2}} < L_D$.

To return to the stratified flow situation, we now derive Long's model under the Boussinesq approximation.

5. The Boussinesq approximation to Long's model

The vorticity equation for a two-dimensional inviscid incompressible flow is

$$\left.\begin{aligned} \mathbf{j}\frac{D}{Dt}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) &= \frac{1}{\rho^2}\nabla\rho\times\nabla p \\ &= -\frac{1}{\rho}\nabla\rho\times(\mathbf{a}+g\mathbf{k}) \\ &= -\frac{1}{\rho}\frac{d\rho}{d\psi}\nabla\psi\times(\mathbf{a}+g\mathbf{k}), \end{aligned}\right\} \tag{16}$$

where $\mathbf{a} = D\mathbf{v}/Dt$. If $|ua_x|$, $|wa_z| \ll |wg|$, which is the essence of the Boussinesq approximation for steady flow, (16) reduces to

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) = \frac{1}{\rho}\frac{d\rho}{d\psi}g\frac{Dz}{Dt}. \tag{17}$$

We note that, in the more usual Boussinesq approach, ρ is replaced by ρ_0 in the denominator on the right-hand side of (17). Following Long (1953), we integrate the above equation along a streamline and find

$$\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{d\rho}{d\psi}gz = F(\psi).$$

If, far upstream,

$$\psi' \to -Uz_0,$$

$$\rho' \to \rho_0 e^{-\beta z_0} = \rho_0 e^{\beta\psi'/U},$$

then the governing equation for the disturbance stream function becomes

$$\nabla^2\psi+\alpha^2\psi = 0, \tag{18}$$

with $\psi \to 0$ as $x \to -\infty$. Under the more usual Boussinesq approach, linear stratification and uniform flow far upstream also lead to (18). Thus, under the Boussinesq approximation the disturbance to a uniform linearly stratified flow is governed by (18) without restriction on the magnitude of the disturbance save that it vanish far upstream; our vortex and doublet solutions do satisfy this requirement. Under Long's model (without the Boussinesq approximation) an upstream condition requiring linear stratification with $u \to U/(1-\beta z_0)^{\frac{1}{2}}$ leads to (18) with the displacement function as the dependent variable. The Boussinesq approximation to Long's model then changes the upstream velocity condition to one of uniform flow. It is in this sense that our vortex and doublet solutions are solutions to Long's model under the Boussinesq approximation.

6. Conclusions

Solutions for line singularities in an unbounded, uniform, linearly stratified flow have been determined. Those singularities across which $L\psi$ is conserved produce no columnar disturbances; doublet and vortex singularities, of this type, have been studied, including the pressure distribution about and drag of a doublet. The vortex singularity may model flow past simple topography in the β-plane.

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